A Stable Robust Optimization Approach for the Portfolio Execution Cost Problem

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Introduction

The portfolio execution cost problem is to minimize the total cost and risk of executing a portfolio of risky assets in several periods. Execution costs are defined by price impact functions, whose estimation is rather challenging and likely erroneous. These estimation errors may severely affect the optimal strategy and the efficient frontier. This motivates the need for a robust approach. However, the common practice of min-max robust optimization might be unstable with respect to changes in the uncertainty set. We propose a novel regularized robust optimization approach for the problem and study its implications.

Portfolio Execution Cost Problem

- Trading large volumes impacts the prices in two ways:

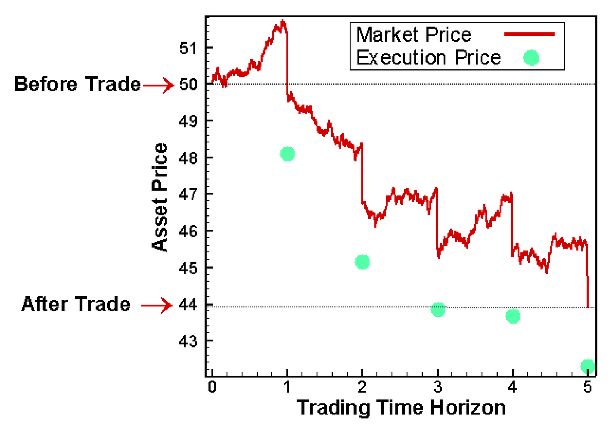

Figure 1: Effect of (large) trades on market prices and execution prices during the course of trading.

Execution Price Dynamic $\tilde{P}_k = P_{k-1} - \frac{H}{\tau} n_k$

Market Price Dynamic $P_k = P_{k-1} + \tau^{1/2} \Sigma B_k - G n_k$

n_k : amount traded at period k, τ : time length between two consecutive trades,
Σ : volatility matrix of asset prices, B_k : stochastic Brownian motion.

H : Temporary impact matrix, G : Permanent impact matrix

- For a liquidation of $\bar{S}$ shares of a portfolio of m assets within N periods:

$$\min_{(n_1,n_2,\ldots,n_N)\in\mathcal{X}} \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right) + \mu\cdot\mathbf{Var}\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right) \quad \text{s.t.} \quad \sum_{k=1}^{N} n_k = \bar{S}$$

where $\mathcal{X}$ is the set of feasible trading strategies, and

P_0 : initial price, μ : risk aversion parameter.

- This is a quadratic programming problem. It can be rewritten as

$$\min_{z\in\mathcal{X}} \frac{1}{\tau}\bar{S}^T H\bar{S} + \frac{1}{2}z^T W(H,G,\mu)z + b^T(H,G)z$$

where $z = (\bar{S}-n_1; \bar{S}-n_1-n_2; \ldots; \bar{S}-\sum_{k=1}^{N-1} n_k)^T \in \mathbb{R}^{m(N-1)}$.

- Estimating impact matrices H and G is rather challenging, partly due to data limitations and price dependent strategies. The literature dealing with improved methods to estimate price impact functions is scarce.
- Theoretical analysis suggests that errors in estimating impact matrices might severely affect the optimal trading strategy and the efficient frontier [1]. This effect reduces when the minimum eigenvalue of the Hessian, W, is large.
- One of the principal methods to address data uncertainty is the robust optimization.

Stability in Robust Optimization

- In robust optimization, data uncertainty is described using an uncertainty set which includes all or most possible realizations of the uncertain input parameters.
- Given a (convex and bounded) uncertainty set $\mathcal{U}$, the robust optimization yields a solution that performs "reasonably well" under all possible realizations of the uncertain input parameters (e.g., see [2]).

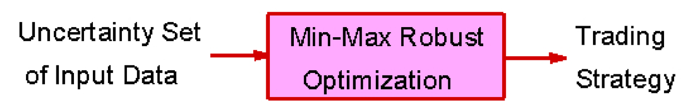

Figure 2: Robust Optimization Scheme

- For the execution cost problem with uncertain impact matrices, the scheme solves

$$\min_{z\in\mathcal{X}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} + \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z.$$

Definition [3]: An optimization scheme to handle deterministic uncertainty is called stable w.r.t. the set-distance metric $d(.,.)$, if given some $\epsilon > 0$,

$$\exists\beta > 0 \quad \text{s.t.} \quad \|x-y\|_2 < \beta, \qquad \forall\hat{\mathcal{U}} \quad \text{s.t.} \quad d(\hat{\mathcal{U}},\mathcal{U}) < \epsilon.$$

where x and y are the optimal solutions obtained from the optimization scheme corresponding to the uncertainty sets $\mathcal{U}$ and $\hat{\mathcal{U}}$, respectively.

Robust optimization scheme might be unstable!

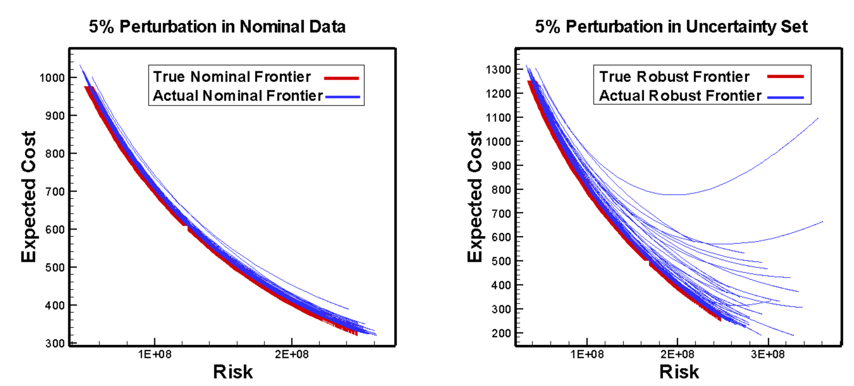

Figure 3: Comparing the sensitivity of the efficient frontier to perturbations in the nominal impact matrices versus perturbations in the boundaries of the uncertainty sets $\mathcal{U}$ of the Impact Matrices.

Main Idea: Regularized Robust Optimization

- Given an uncertainty set $\mathcal{U}$, construct a regularized uncertainty set by including an appropriate regularization constraint. This regularized uncertainty set is then used in the min-max robust optimization scheme.

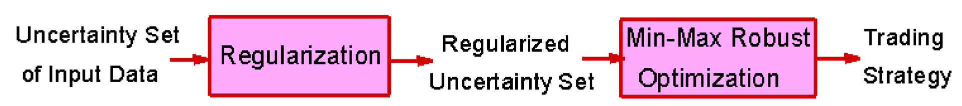

Figure 2: Regularized Robust Optimization Scheme

- For the execution cost problem with uncertain impact matrices, given an uncertainty set $\mathcal{U}$ and the regularization parameter $\rho > 0$, we construct the regularized uncertainty set:

$$\mathcal{U}_\rho = \left\{(\tilde{H},\tilde{G}) \in \mathcal{U} \mid \lambda_{\min}(W(\tilde{H},\tilde{G},0)) \geq \rho\right\}$$

- The regularized uncertainty set preserves the convexity of the uncertainty set $\mathcal{U}$.

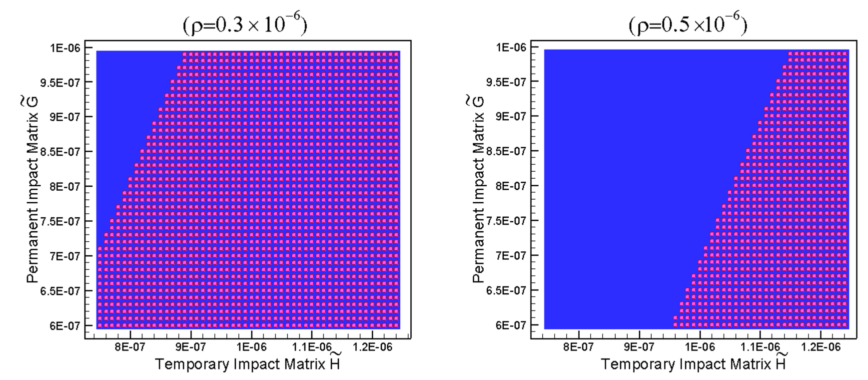

Figure 2: Regularized uncertainty set $\mathcal{U}_\rho$ (pink area) versus the uncertainty set $\mathcal{U}$ (blue area) for a single asset trading.

It can be shown that regularized robust optimization is stable!

Implications of the Proposed Approach

- An optimal strategy of the regularized robust optimization approach is found by solving

$$\min_{z\in\mathcal{X}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}_\rho} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} + \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z.$$

- When $\mathcal{X}$ is convex and $W \succeq 0$, the regularized robust formulation is a convex problem.
- When $\mathcal{X}$ is compact and convex, the min-max problem has a unique saddle point.
- The regularized robust formulation can be solved using an interior-point method [4].
- Theorem: The difference between the optimal trading strategy n^* of the regularized robust optimization scheme and the naive strategy $n_k = \frac{\bar{S}}{N}$ satisfies

$$\left\|n^* - \frac{\bar{S}}{N}\right\|_2 \leq \left(\left(\frac{\nu}{\rho + 2\mu\tau\lambda_{\min}(\Sigma\Sigma^T)}\right)^2 - \frac{1}{N}\right)^{1/2} \|\bar{S}\|_2,$$

where the constant $\nu > 0$ is determined by N and the uncertainty set $\mathcal{U}$.

The diversification of the optimal strategy can be adjusted by ρ!

- Theorem: Let ρ_1 and ρ_2 be two regularization parameters where $\rho_1 \leq \rho_2$. Then the robust frontier corresponding to ρ_2 is always settled below the one corresponding to ρ_1.

The conservativeness of the optimal strategy can be adjusted by ρ!

- Theorem: The objective value of the trading strategy n^* when the impact matrices do not lie in $\mathcal{U}_\rho$ is less than the robust optimal value with probability at least Pr $\left(\lambda_{\min}(\tilde{W}) \geq \rho\right)$.

Conclusion and Future Work

In this work, we propose a regularized robust scheme that provides stable solutions w.r.t. perturbations in the uncertainty set. The approach allows controlling the diversification and conservativeness of the solution. It ensures deterministic and probabilistic guarantees on the objective value as data change. The probabilistic guarantee is independent of the obtained solution. For future work, it remains to investigate the impact of the regularization parameter on the total amount selling short, when it is allowed.

References

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